More complex oscillations of systems may be used as references, for which it is possible to determine the viscoelastic properties and its density by a combination of calculation formulas.

Similarly, it is possible using (17), to obtain expressions for the medium characteristics, when the medium surrounds the shell, and the oscillations are recorded on its inner surface.

The proposed principle can also be extended to some other shell forms.

For the practical realization of the proposed method it is necessary to know the displacements and stresses on the observation surface S_1 and process the data obtained using the formulas proposed above.

Analog and discrete systems of three-dimensional processing have found wide application in acoustic measurements /5, 7/. An example of the use of a discrete system is the set of transducers of displacement (velocity, acceleration) on the surface of a technological apparatus shell (the acceptable pitch transducers is determined using Kotel'nikov's theorem). The displacement pickup (velocities, accelerations) and stresses (pickup elements of strain gauge) may alternate and a concurrent measurement of stresses and displacements does not cause any difficulties. Further data processing can be carried out on simple computing equipment.

Since the proposed method does not require the oscillations to be of any specific form, it is possible to excite the shell by a priori specified stresses (e.g., application of a point force) and measure only displacements.

In the analog form of the measurement system it is possible to use electromechanical transducers located around the shell and performing direct integration in analog form of shell displacements by the summation of emfs, currents, charges, magnetic fluxes, etc.

Note that since the form of the oscillations is arbitrary, it is possible to excite in the shell oscillations that decay rapidly with distance (non-uniform waves), while at the same time reducing the observation surface.

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REPRESENTATION IN TERMS OF p-ANALYTIC FUNCTIONS OF THE GENERAL SOLUTION OF EQUATIONS OF THE THEORY OF ELASTICITY OF A TRANSVERSELY ISOTROPIC BODY*

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A general solution is given for the equations of the theory of elasticity in terms of p-analytic functions for a transversely isotropic body in a non-axisymmetric stress state. This representation was obtained in /1/ for an isotropic medium. For the transport medium a similar representation is known only for the axisymmetric problem /2—4/.

1. We shall call the function

$$f(z, r) = p(z, r) + iq(z, r) \equiv \binom{p}{q}_\alpha$$

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(r^k, α) -analytic (or (k, α) -analytic), if p and q satisfy the system

$$\frac{\partial p}{\partial r} = \frac{\alpha}{r^k}\frac{\partial q}{\partial z}, \quad \frac{\partial p}{\partial z} = -\frac{1}{r^k\alpha}\frac{\partial q}{\partial r}, \quad \alpha > 0 \tag{1.1}$$

Although by a change of scale by one variable, for example $z = \alpha\zeta$, the (k, α) -analytic function $f(z, r)$ can be reduced to the conventional r^k-analytic function $f(\zeta, r)$ /5/, there is an undoubted advantage in using (k, α)-analytic functions directly, as will be shown below.

For work with such functions it is advantageous to introduce metric differential operators

$$\overline{M}_k{}^\alpha = \begin{Vmatrix} r^k\frac{\partial}{\partial r} & -\alpha\frac{\partial}{\partial z} \\ \alpha\frac{\partial}{\partial z} & \frac{1}{r^k}\frac{\partial}{\partial r} \end{Vmatrix}, \quad M_k{}^\alpha = \begin{Vmatrix} \frac{\partial}{\partial r} & \alpha\frac{\partial}{\partial z}r^k \\ -\alpha\frac{\partial}{\partial z}\frac{1}{r^k} & \frac{\partial}{\partial r} \end{Vmatrix}$$

$$\overline{K}_k{}^\alpha = \begin{Vmatrix} r^k\frac{\partial}{\partial r} & \alpha\frac{\partial}{\partial z} \\ -\alpha\frac{\partial}{\partial z} & \frac{1}{r^k}\frac{\partial}{\partial r} \end{Vmatrix}, \quad K_k{}^\alpha = \begin{Vmatrix} \frac{\partial}{\partial r} & -\alpha\frac{\partial}{\partial z}r^k \\ \alpha\frac{\partial}{\partial z}\frac{1}{r^k} & \frac{\partial}{\partial r} \end{Vmatrix}$$

Using these operators, the conditions of (k, α)-analyticity (1.1), of $(-k, \alpha)$-analyticity, of (k, α)-anti-analytisity and $(-k,\alpha)$-anti-analyticity, of the function $p + iq$ can be written, respectively, in the form

$$\overline{M}_k{}^\alpha\begin{pmatrix} p \\ q \end{pmatrix} = 0, \quad K_k{}^\alpha\begin{pmatrix} p \\ q \end{pmatrix} = 0, \quad \overline{K}_k{}^\alpha\begin{pmatrix} p \\ q \end{pmatrix} = 0, \quad M_k{}^\alpha\begin{pmatrix} p \\ q \end{pmatrix} = 0$$

The properties of these functions and operators are similar to those in /1/. For example, the general solution of equation

$$M_k{}^\alpha\overline{M}_k{}^\alpha\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \alpha^2\frac{\partial}{\partial z}\left(r^k\frac{\partial p}{\partial z}\right) + \frac{\partial}{\partial r}\left(r^k\frac{\partial p}{\partial r}\right) \\ \alpha^2\frac{\partial}{\partial z}\left(\frac{1}{r^k}\frac{\partial q}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r^k}\frac{\partial q}{\partial z}\right) \end{pmatrix} = 0 \tag{1.2}$$

is the sum of the (k, α)-analytic function $\varphi + i\psi$ and of the (k, α)-anti-analytic function $\Phi - i\Psi$.

2. We will use the equations of the theory of elasticity for an isothermal transport medium in displacements /6/, and for further purposes we will write them in the form

$$\frac{\tau}{A_{44}}\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + N\frac{\partial}{\partial z}\left(M\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r}\right) = 0 \tag{2.1}$$

$$\beta^2\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta} + R\frac{\partial\Omega}{\partial r} = 0 \tag{2.2}$$

$$\beta^2\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta} + R\frac{1}{r}\frac{\partial\Omega}{\partial\theta} = 0 \tag{2.3}$$

$$\Omega = S\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r}$$

$$\beta^2 = \frac{2A_{44}}{A_{11} - A_{12}}, \quad N = \frac{A_{13} + A_{44}}{A_{44}}, \quad M = \frac{A_{33} - \tau}{A_{13} + A_{44}}$$

$$R = \frac{A_{11} + A_{12}}{A_{11} - A_{12}}, \quad S = \frac{2(A_{13} + A_{44})}{A_{11} + A_{12}}$$

where w, u, v are the axial, radial and tangential displacements in the cylindrical coordinate system (z, r, θ), τ is an, as yet, arbitrary parameter, A_{ij} are the moduli of elasticity, and it is assumed that $\beta^2 > 0$.

Eliminating from (2.2) and (2.3) the quantity Ω , we will show that the function

$$\omega = \frac{\partial u}{\partial\theta} - \frac{\partial}{\partial r}(rv)$$

satisfies the equation

$$\beta^2\frac{\partial^2\omega}{\partial z^2} + \frac{\partial^2\omega}{\partial r^2} - \frac{1}{r}\frac{\partial\omega}{\partial r} + \frac{\omega}{r^2} + \frac{1}{r^2}\frac{\partial^2\omega}{\partial\theta^2} = 0 \tag{2.4}$$

Let us assume that w, u and v can be represented by the Fourier series

$$w = w_0{}^1 + \sum_{n=1} w_n{}^1\cos n\theta + w_n{}^2\sin n\theta$$

$$u = u_0{}^1 + \sum_{n=1} u_n{}^1\cos n\theta + u_n{}^2\sin n\theta$$

$$v = v_0{}^2 + \sum_{n=1} -v_n{}^1\sin n\theta + v_n{}^2\cos n\theta$$

Then for $n = 0$ we obtain the system

$$\frac{\tau}{A_{44}}\frac{\partial^2 w_0^1}{\partial z^2} + \frac{\partial^2 w_0^1}{\partial r^2} + \frac{1}{r}\frac{\partial w_0^1}{\partial r} + N\frac{\partial}{\partial z}\left(M\frac{\partial w_0^1}{\partial z} + \frac{\partial u_0^1}{\partial r} + \frac{u_0^1}{r}\right) = 0 \quad (2.5)$$

$$\beta^2\frac{\partial^2 u_0^1}{\partial z^2} + \frac{\partial^2 u_0^1}{\partial r^2} + \frac{1}{r}\frac{\partial u_0^1}{\partial r} - \frac{1}{r^2}u_0^1 + R\frac{\partial \Omega_0^1}{\partial r} = 0 \quad (2.6)$$

$$\beta^2\frac{\partial^2 v_0^2}{\partial z^2} + \frac{\partial^2 v_0^2}{\partial r^2} + \frac{1}{r}\frac{\partial v_0^2}{\partial r} - \frac{v_0^2}{r^2} = 0 \quad (2.7)$$

and for $n \geqslant 1$ we have the system

$$\frac{\tau}{A_{44}}\frac{\partial^2 w_n}{\partial z^2} + \frac{\partial^2 w_n}{\partial r^2} + \frac{1}{r}\frac{\partial w_n}{\partial r} - \frac{n^2}{r^2}w_n + N\frac{\partial}{\partial z}\left(M\frac{\partial w_n}{\partial z} + \frac{\partial u_n}{\partial r} + \frac{u_n - nv_n}{r}\right) = 0 \quad (2.8)$$

$$\beta^2\frac{\partial^2 u_n}{\partial z^2} + \frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r} - \frac{n^2+1}{r^2}u_n + \frac{2n}{r^2}v_n + R\frac{\partial \Omega_n}{\partial r} = 0 \quad (2.9)$$

$$\beta^2\frac{\partial^2 v_n}{\partial z^2} + \frac{\partial^2 v_n}{\partial r^2} + \frac{1}{r}\frac{\partial v_n}{\partial r} - \frac{n^2+1}{r^2}v_n + \frac{2n}{r^2}u_n + R\frac{n}{r}\Omega_n = 0 \quad (2.10)$$

$$\Omega_n = S\frac{\partial w_n}{\partial z} + \frac{\partial u_n}{\partial r} + \frac{u_n - nv_n}{r}$$

The superscripts in (2.8)–(2.10) are omitted, since these equations are the same for both superscripts.

We will convert system (2.8)–(2.10) to a form that is more convenient for using matrix operators. We put

$$U_n = u_n - v_n, \quad V_n = u_n + v_n$$

Subtracting and adding (2.9) and (2.10), we obtain the equations

$$\left(\beta^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(n+1)^2}{r^2}\right)U_n = -R\left(\frac{\partial}{\partial r} - \frac{n}{r}\right)\Omega_n \quad (2.11)$$

$$\left(\beta^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(n-1)^2}{r^2}\right)V_n = -R\left(\frac{\partial}{\partial r} + \frac{n}{r}\right)\Omega_n \quad (2.12)$$

It can be directly verified that

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(n+1)^2}{r^2}\right)U_n = T_n^+ \quad (2.13)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(n-1)^2}{r^2}\right)V_n = T_n^- \quad (2.14)$$

$$T_n^\pm = \left(\frac{\partial}{\partial r} \mp \frac{n}{r}\right)\mu_n \pm \frac{1}{r}\left(\frac{\partial}{\partial r} \mp \frac{n \pm 1}{r}\right)\omega_n$$

$$\mu_n = \frac{\partial u_n}{\partial r} + \frac{u_n - nv_n}{r}, \quad \omega_n = nu_n - \frac{\partial}{\partial r}(rv_n)$$

where ω_n is one of the coefficients of the Fourier series of ω. It follows from (2.4) that ω_n satisfies the equation

$$\beta^2\frac{\partial^2 \omega_n}{\partial z^2} + \frac{\partial^2 \omega_n}{\partial r^2} - \frac{1}{r}\frac{\partial \omega_n}{\partial r} - \frac{n^2-1}{r^2}\omega_n = 0 \quad (2.15)$$

We multiply formula (2.13) by the coefficient η and add it to (2.11), and carry out a similar operation on (2.14) and (2.12). After dividing the results obtained by $(1+\eta)$, we have

$$\frac{\beta^2}{1+\eta}\frac{\partial^2 U_n}{\partial z^2} + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(n+1)^2}{r^2}\right)U_n = L_n^+ \quad (2.16)$$

$$\frac{\beta^2}{1+\eta}\frac{\partial^2 V_n}{\partial z^2} + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(n-1)^2}{r^2}\right)V_n = L_n^- \quad (2.17)$$

$$L_n^\pm = -\frac{R-\eta}{1+\eta}\left(\frac{\partial}{\partial r} \mp \frac{n}{r}\right)\left(\frac{RS}{R-\eta}\frac{\partial w_n}{\partial r} + \frac{\partial u_n}{\partial r} + \frac{u_n - nv_n}{r}\right) \pm \frac{\eta}{1+\eta}\frac{1}{r}\left(\frac{\partial}{\partial r} \mp \frac{n \pm 1}{r}\right)\omega_n$$

Inspection of (2.16) and (2.17) together with (2.8) shows that the two parameters τ and η can be selected so that the equations

$$\frac{\beta^2}{1+\eta} = \frac{\tau}{A_{44}}, \quad \frac{RS}{R-\eta} = \frac{A_{33}-\tau}{A_{13}+A_{44}}$$

are satisfied. This results in the following equation for τ:

$$A_{11}\tau^2 - (A_{11}A_{33} - A_{13}^2 - 2A_{13}A_{44})\tau + A_{33}A_{44}^2 = 0 \quad (2.18)$$

The roots of this equation τ_1 and τ_2 can be real or complex /6/, and $\sqrt{\tau_1}$ and $\sqrt{\tau_2}$ cannot be purely imaginary. We shall, therefore, assume that at least one root of (2.18) is positive, and shall use it subsequently.

Note that for an isotropic material $\tau_1 = \tau_2 = 1$, so that for a "not strongly" anisotropic material the roots of (2.18) can be both taken as positive; they are both positive, provided that

$$A_{11}A_{33} - A_{13}^2 - 2A_{13}A_{44} > 2A_{44}\sqrt{A_{11}A_{33}}, \quad A_{44} > 0$$

Using the positive root of (2.18), we reduce (2.8), (2.16) and (2.17) to the form

$$\Delta_n^\alpha W_n + lk\alpha^{-2}\frac{\partial\vartheta_n}{\partial z} = 0 \tag{2.19}$$

$$\Delta_{n+1}^\alpha U_n + l\left(\frac{\partial}{\partial r} - \frac{n}{r}\right)\vartheta_n = \frac{\eta}{1+\eta}\frac{1}{r}\left(\frac{\partial}{\partial r} - \frac{n+1}{r}\right)\omega_n \tag{2.20}$$

$$\Delta_{n-1}^\alpha V_n + l\left(\frac{\partial}{\partial r} + \frac{n}{r}\right)\vartheta_n = -\frac{\eta}{1+\eta}\frac{1}{r}\left(\frac{\partial}{\partial r} + \frac{n-1}{r}\right)\omega_n \tag{2.21}$$

where

$$\vartheta_n = k\frac{\partial w_n}{\partial z} + \frac{\partial u_n}{\partial r} + \frac{u_n - nv_n}{r}$$

$$\Delta_n^\alpha = \alpha^2\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}$$

$$\alpha^2 = \frac{\tau}{A_{44}}, \quad l = \frac{A_{11}\tau - A_{44}^2}{A_{44}^2}, \quad k = \frac{A_{33} - \tau}{A_{13} + A_{44}}, \quad \eta = \frac{\beta^2 A_{44}}{\tau} - 1$$

From (2.19) — (2.21) we can establish that ϑ_n satisfies the equation

$$\Delta_n^\gamma\vartheta_n = 0; \quad \gamma^2 = \frac{\alpha^4 + lk^2}{\alpha^2(1+l)} \tag{2.22}$$

We further assume that $\gamma^2 \neq \alpha^2$, $\beta^2 \neq \alpha^2$, and $\beta^2 \neq \gamma^2$.

3. Using the substitutions

$$w_n = r^n Z_n, \quad \vartheta_n = r^n\Theta_n, \quad U_n = r^{-(n+1)}Y_n, \quad \omega_n = r^{n+1}a_n$$

we reduce (2.19) and (2.20) to the form

$$M_{2n+1}^\alpha \overline{M}_{2n+1}^\alpha\begin{pmatrix} Z_n \\ Y_n\end{pmatrix} = -l\begin{pmatrix} k\alpha^{-2}r^{2n+1}\dfrac{\partial\Theta_n}{\partial z} \\ \dfrac{\partial\Theta_n}{\partial r}\end{pmatrix} + \frac{\eta}{1+\eta}\begin{pmatrix} 0 \\ \dfrac{\partial a_n}{\partial r}\end{pmatrix} \tag{3.1}$$

in which by virtue of (2.22) the function Θ_n is $(2n+1, \gamma)$-harmonic, and by virtue of (2.15) a_n is $(2n+1, \beta)$-harmonic.

The solution of (3.1) is constructed in almost the same way as that of an isotropic medium /1/, and has the form

$$\begin{pmatrix} Z_n \\ Y_n\end{pmatrix} = \begin{pmatrix}\Phi_n \\ \Psi_n\end{pmatrix}_\alpha + \frac{1}{2\alpha}\begin{pmatrix} P_{0n} \\ -Q_{0n}\end{pmatrix}_\alpha + \begin{pmatrix} AP_n \\ BQ_n\end{pmatrix}_\gamma + \begin{pmatrix} A_1P_{1n} \\ B_0Q_{1n}\end{pmatrix}_\beta \tag{3.2}$$

$$\left(A = \frac{lk}{\alpha^2(\gamma^2 - \alpha^2)}, \quad B = \frac{l\gamma}{\gamma^2 - \alpha^2}, \quad A_1 = 0, \quad B_1 = \frac{\eta}{1+\eta}\frac{\beta}{\alpha^2 - \beta^2}\right)$$

where the first two terms according to (1.2) represent the solution of the homogeneous system (3.1) (and $\Phi_n + i\Psi_n$ and $P_{0n} + iQ_{0n}$ are arbitrary $(2n+1, \alpha)$-analytic functions); the third term is a particular solution of (3.1) when $a_n = 0$ and for arbitrary Θ_n of this class; the last term represents a particular solution for $\Theta_n = 0$ and arbitrary a_n.

For a $(2n+1, \gamma)$-analytic function $P_n + iQ_n$ the following condition is satisfied:

$$dP_n/dz = \Theta_n \tag{3.3}$$

and for a $(2n+1, \beta)$-analytic function $P_{1n} + iQ_{1n}$ the condition

$$dP_{1n}/dz = a_n \tag{3.4}$$

is satisfied.

Note that for an isotropic medium $\alpha = \gamma = 1$, and a solution of the form (3.2) does not exist. This is related to the difference in the representation of the solutions for the transtropic and isotropic media, as shown in /1/.

Introducing the substitutions

$$w_n = r^{-n}Z_n^*,\ \vartheta_n = r^{-n}\Theta_n^*,\ V_n = r^{n-1}Y_n^*,\ \omega_n = r^{-n+1}b_n^*$$

(2.21) and (2.19) can be written in the form

$$K^{\alpha}_{2n-1}\overline{K}^{\alpha}_{2n-1}\begin{pmatrix}Y_n^*\\ Z_n^*\end{pmatrix} = -l\begin{pmatrix}\dfrac{\partial\Theta_n}{\partial r}\\ k\alpha^{-2}r^{-2n+1}\dfrac{\partial\Theta_n^*}{\partial z}\end{pmatrix} - \frac{\eta}{1+\eta}\begin{pmatrix}\dfrac{\partial b_n^*}{\partial r}\\ 0\end{pmatrix} \tag{3.5}$$

where Θ_n^* is $(-2n+1, \gamma)$-harmonic, and b_n^* $(-2n+1, \beta)$-harmonic. The solution of (3.5) in which Θ_n^* and b_n^* are so far regarded as arbitrary functions, have the form

$$\begin{pmatrix}Y_n^*\\ Z_n^*\end{pmatrix} = \begin{pmatrix}\Phi_n^*\\ -\Psi_n^*\end{pmatrix}_{\alpha} - \frac{1}{2\alpha}\begin{pmatrix}P_{0n}^*\\ Q_{0n}^*\end{pmatrix}_{\alpha} + \begin{pmatrix}-BP_n^*\\ AQ_n^*\end{pmatrix}_{\gamma} + \begin{pmatrix}B_1P_{1n}^*\\ -A_1Q_{1n}^*\end{pmatrix}_{\beta} \tag{3.6}$$

where the functions $\Phi_n^* + i\Psi_n^*$ and $P_{0n}^* + iQ_{0n}^*$ are $(2n-1, \alpha)$-analytic, and $P_n^* + iQ_n^*$ is $(2n-1, \gamma)$-analytic, and

$$dQ_n^*/dz = \Theta_n^* \tag{3.7}$$

and function $P_{1n}^* + iQ_{1n}^*$ is $(2n-1, \beta)$-analytic and

$$dQ_{1n}^*/dz = b_n^* \tag{3.8}$$

When (3.3), (3.4), (3.7), and (3.8) are satisfied, (3.2) and (3.6) may be considered as the general solution of (2.19)−(2.21) for arbitrary functions ϑ_n and ω_n. If these functions are to provide the general solution of the theory of elasticity, the identities

$$k\frac{\partial w_n}{\partial z} + \frac{\partial u_n}{\partial r} + \frac{u_n - nv_n}{r} = \vartheta_n = r^n\Theta_n = r^n\frac{\partial P_n}{\partial z} \tag{3.9}$$
$$nu_n - \frac{\partial}{\partial r}(rv_n) = \omega_n = r^{n+1}a_n = r^{n+1}\frac{\partial P_{1n}}{\partial z}$$

must be satisfied.

In addition, the functions considered are connected by three more relations

$$Q_n^* = r^{2n}P_n,\ Q_{1n}^* = r^{2n}P_{1n},\ Z_n^* = r^{2n}Z_n \tag{3.10}$$

The first of these follows from (3.3) and (3.7), the second from (3.4) and (3.8), and the third from the formulas for w_n.)

The presence of the five conditions (3.9) and (3.10) in formulas (3.2) and (3.6) reduces the arbitrariness from eight to three functions. Using the above conditions, we can obtain

$$\begin{pmatrix}P_{0n}\\ Q_{0n}\end{pmatrix} = C\begin{pmatrix}\Phi_n\\ \Psi_n\end{pmatrix};\quad \begin{pmatrix}P_{0n}^*\\ Q_{0n}^*\end{pmatrix} = C\begin{pmatrix}\Phi_n^*\\ \Psi_n^*\end{pmatrix};\quad C = 2\alpha\frac{\alpha-k}{\alpha+k} \tag{3.11}$$

$$\Psi_n^* = -r^{2n}\Phi_n \tag{3.12}$$

The final solution of (2.19)−(2.21) can be written in the form

$$\begin{pmatrix}Z_n\\ Y_n\end{pmatrix} = \begin{pmatrix}\Phi_n\\ \Psi_n\end{pmatrix}_{\alpha} + \frac{\alpha-k}{\alpha+k}\begin{pmatrix}\Phi_n\\ -\Psi_n\end{pmatrix}_{\alpha} + \begin{pmatrix}AP_n\\ BQ_n\end{pmatrix}_{\gamma} + \begin{pmatrix}A_1P_{1n}\\ B_1Q_{1n}\end{pmatrix}_{\beta} \tag{3.13}$$

$$\begin{pmatrix}Y_n^*\\ Z_n^*\end{pmatrix} = \begin{pmatrix}\Phi_n^*\\ -\Psi_n^*\end{pmatrix}_{\alpha} - \frac{\alpha-k}{\alpha+k}\begin{pmatrix}\Phi_n^*\\ \Psi_n^*\end{pmatrix}_{\alpha} + \begin{pmatrix}-BP_n^*\\ AQ_n^*\end{pmatrix}_{\gamma} + \begin{pmatrix}B_1P_{1n}^*\\ -A_1Q_{1n}^*\end{pmatrix}_{\beta} \tag{3.14}$$

All three functions $\Phi_n + i\Psi_n$, $P_n + iQ_n$ and $P_{1n} + iQ_{1n}$ in (3.13) can be regarded as arbitrary; the remaining functions can be expressed in terms of these three, because the first two formulas (3.10) and (3.12) define the relation between the functions of the second row of formula (3.14), and the functions of the first row of formula (3.13).

For the axisymmetric problem (2.5), (2.6) the solution has the form

$$\begin{pmatrix}w_0^1\\ ru_0^1\end{pmatrix} = \begin{pmatrix}\Phi_0\\ \Psi_0\end{pmatrix}_{\alpha} + \frac{\alpha-k}{\alpha+k}\begin{pmatrix}\Phi_0\\ -\Psi_0\end{pmatrix}_{\alpha} + \begin{pmatrix}AP_0\\ BQ_0\end{pmatrix}_{\gamma}$$

which is equivalent to the results in /2/.

The general solution of (2.7) that defines the twisting of a transtropic medium may be represented either in the form $v_0^2 = r\Phi_0$, where Φ_0 is a $(3, \beta)$-harmonic function, or in the form $v_0^2 = r^{-1}\Psi_0^*$, where Ψ_0^* is a $(-1, \beta)$-harmonic function.

As in the isotropic case /1/ we can introduce three $(2n+1)$-harmonic functions that with explicitly express all interconnected functions of (3.13) and (3.14). For this we introduce the $(2n+1,\ \alpha)$-harmonic function φ_n, the $(2n+1,\ \gamma)$-harmonic function Ψ_n, and the $(2n+1,\ \beta)$-harmonic function χ_n for which

$$\Phi_n = \alpha\frac{\partial\varphi_n}{\partial z},\quad \Psi_n = r^{2n+1}\frac{\partial\varphi_n}{\partial r},\quad \Phi_n{}^* = r\frac{\partial\varphi_n}{\partial r} + 2n\varphi_n,\quad \Psi_n{}^* = -r^{2n}\alpha\frac{\partial\varphi_n}{\partial z}$$

$$P_n = \gamma\frac{\partial\psi_n}{\partial z},\quad Q_n = r^{2n+1}\frac{\partial\psi_n}{\partial r},\quad P_n{}^* = -r\frac{\partial\psi_n}{\partial r} - 2n\psi_n,\quad Q_n{}^* = r^{2n}\gamma\frac{\partial\psi_n}{\partial z}$$

$$P_{1n} = \beta\frac{\partial\chi_n}{\partial z},\quad Q_{1n} = r^{2n+1}\frac{\partial\chi_n}{\partial r},\quad P_{1n} = -r\frac{\partial\chi_n}{\partial r} - 2n\chi_n,\quad Q_{1n} = r^{2n}\beta\frac{\partial\chi_n}{\partial z}$$

The displacements can be expressed in terms of the functions introduced as follows:

$$w_n = r^n\frac{\partial}{\partial z}\left(\frac{2\alpha^2}{\alpha+k}\varphi_n + A\gamma\psi_n\right) \tag{3.15}$$

$$u_n - v_n = r^n\frac{\partial}{\partial r}\left(\frac{2k}{\alpha+k}\varphi_n + B\psi_n + B_1\chi_n\right)$$

$$u_n + v_n = r^{-n}\frac{\partial}{\partial r}r^{2n}\left(\frac{2k}{\alpha+k}\varphi_n + B\psi_n + B_1\chi_n\right)$$

The representations (3.13), (3.14), or (3.15) may be considered as an analog of the Kolosov-Muskhelishvili formulas for the three-dimensional stress state of a transversely isotropic medium.

We note in conclusion that all of the formulas derived remain valid when the roots of Eq.(2.18) are complex. It is only necessary to introduce into consideration $(r^k,\ \alpha)$-analytic functions with complex constants α.

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SPECTRAL RELATIONSHIPS FOR THE INTEGRAL OPERATORS GENERATED BY A KERNEL IN THE FORM OF A WEBER-SONIN INTEGRAL, AND THEIR APPLICATION TO CONTACT PROBLEMS *

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Generalized potential theory methods are used to re-establish the spectral relationship /1/ for the integral operators generated by a symmetric kernel in the form of the Weber-Sonin integral in the finite interval $(0,\ a)$, the kernel containing Jacobi polynomials. Spectral relations are also established for the integral operator generated by the same kernel in the semi-infinite interval $(a,\ \infty)$, and other allied relationships. The latter

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